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Characteristic Representation of Elementary Cellular Automata*

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Abstract

We propose a characteristic representation of one-dimensional and 2-state, 3-neighbor cellular automaton rules, which describes an effective form of each rule after many time steps. Simulated results of the representation show that complex structures of Class IV rules come from their aspects of Class II and III. Class IV-like patterns can be generated by characteristic functions of a linear combenation of Class II and III rule functions.

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1 Introduction

Cellular automata (CAs) are discrete dynamical systems with a lattice of sites. Each site has a finite set of possible values which evolve synchronously in discrete time steps according to identical rules. Wolfram[1][2] undertook valuable works to search systematically for interesting phenomena and possible behavior of one-dimensional automata. He found essential four types of behavior: homogeneous (Class I), periodic (Class II), chaotic (Class III) and complex (Class IV). This classification scheme has been discussed using several characteristic parameters[3-11]. In the previous work[11] of one of the authors and collaborators, characteristic parameters are calculated in all rules of the simplest one-dimensional CAs and applied to give quantitative grounds for the Wolfram's phenomenological classification. The simplest CAs consist of a line of sites with each site carrying a value 0 or 1. The values of a particular site is determined by the previous values of its nearest neighbors. They are called one-dimensional and 2-state, 3-neighbor CAs or *elementary* CAs. Rules 54 and 110, following Wolfram's numbering[2], are thought to be in Class IV[6]. Typical patterns generated by them show long transients and soliton-like phenomena(Fig. 1) . According to the previous paper[11], D_{iff}, the spreading rate of difference patterns, is the most powerful parameter to classify the rules. The calculated results are nearly sufficient for categorizing the rules to Class I, II and III. The Class IV rules, however, could not be distinguished from Class III rules. Their results of D_{iff} are tentatively sorted in Table 1. It is remarkable that rules 54 and 110 exist between Class II and III groups. This result supports that the formalism is available for characterizing Class IV rules and we can conjecture that their features come from interference between their aspects of Class II and III. Simultaneously, it represents the confines of discussions on a single characteristic parameter. Because rule 41, for example, has the close value of D_{iff} with those of rules 54 and 110, we cannot distinguish them. So we present a characteristic representation of each rule, which is given by a linear combination of independent rule functions. We can use this formalism to classify CA rules because it describes an effective form of each rule function after many time steps.

2 Characteristic Representation

We take x_k to denote a value of the kth site in a one-dimensional Cellular Automaton. Each site value is specified as an integer 0 or 1 (2-state). If there are N total sites, a configuration can be denoted by

$$X = (x_1, x_2, \ldots, x_N) \ . \tag{2.1}$$

The Hamming distance of X from $\mathbf{0} = (0, \ldots, 0)$ is identical to its *weight*,

$$w(X) = \sum_{k=1}^{N} x_k \ . \tag{2.2}$$

We define a configuration space $\{X\}_N$ which is a set of all distinct configurations represented by the above form. Time evolution of a site value is determined by iteration of the mapping

$$\begin{aligned} x_k^{(t+1)} &= f(x_{k-r}^{(t)}, \ldots, x_k^{(t)}, \ldots, x_{k+r}^{(t)}) \\ &\equiv f_r(x_k^{(t)}), \end{aligned} \tag{2.3}$$

where f is called a *rule function* which specifies the CA. The subscript r determines a neighborhood of the rule function ($(2r+1)$-neighbor). In the following, we abbreviate the subscript because our discussions are restricted to 3-neighbor CAs. A rule function is determined by the succeeding 8 values

$$f(0,0,0), f(0,0,1), \cdots, f(1,1,1), \tag{2.4}$$

or $f(0), f(1), \cdots, f(7)$ for short. There are $2^8 = 256$ rules of the elementary CAs and rule numbers are defined by

$$\sum_{i=0}^{7} f(i) \cdot 2^i. \tag{2.5}$$

Following the notation of the previous paper[11], if $f^{(1)}$ denotes a rule function of the first time step, the function of the tth time step can be obtained by

$$f^{(t)}(x_k) = f^{(1)}(f^{(1)}(\cdots f^{(1)}(x_k)\cdots)), \tag{2.6}$$

which is a rule for $(2t+1)$-neighbor of x_k:

$$f^{(t)}(x_k) = f^{(t)}(x_{k-t}, \ldots, x_k, \ldots, x_{k+t}) \ . \tag{2.7}$$

By the tth rule function we can define a mapping on the configuration space $\{X\}_N$, $\boldsymbol{f}^{(t)} : \{X\}_N \to \{X\}_N$ such as

$$\begin{aligned} \boldsymbol{f}^{(t)}(X) &= \boldsymbol{f}^{(t)}((x_1, x_2, \dots, x_N)) \\ &= (f^{(t)}(x_1), f^{(t)}(x_2), \dots, f^{(t)}(x_N)), \end{aligned} \tag{2.8}$$

with a proper boundary treatment. In this article, we adopt the periodic boundary condition. The image of $\boldsymbol{f}^{(t)}$, $\boldsymbol{f}^{(t)}(\{X\}_N)$, is a subset of $\{X\}_N$. Since the following is a monotone decreasing sequence of $\boldsymbol{f}^{(t)}(\{X\}_N)$'s,

$$\{X\}_N \supset \boldsymbol{f}^{(1)}(\{X\}_N) \supset \boldsymbol{f}^{(2)}(\{X\}_N) \supset \cdots \supset \boldsymbol{f}^{(t)}(\{X\}_N) \supset \cdots\ , \tag{2.9}$$

there exists a limit set $\lim\limits_{t\to\infty} \boldsymbol{f}^{(t)}(\{X\}_N)$, which will be thought to contain characteristics of each CA because its asymptotic behavior is fixed to an element or a subset of this set. Classification of CA rules will be carried out through arrangement of rule functions by similarity of their limit sets. If limit sets of two different rules are equivalent or similar for sufficiently large total sites, we can recognize these rule functions to be effectively the same, which suggests the existence of a representation of rule functions taking the similarity of limit sets into account.

We take $f_R^{(1)}$ to denote the rule function of rule R of the elementary CAs. After t time evolutions, the domain of the mapping $\boldsymbol{f}_R^{(1)}$ to the $(t+1)$th time step is $\boldsymbol{f}_R^{(t)}(\{X\}_N)$. Because $\boldsymbol{f}_R^{(t)} : \{X\}_N \to \{X\}_N$ is a many-to-one mapping in general, we define multiplicity of the mapping for an element $X^{(t)}$ of $\boldsymbol{f}_R^{(t)}(\{X\}_N)$ as $m_R\left(X^{(t)}\right)$. Moreover $n_R\left(i, X^{(t)}\right)$ denotes how many times each configuration of three sites, $(0,0,0), (0,0,1), \cdots, (1,1,1)$ ($i = 0, 1, \dots, 7$, respectively), appears in a configuration $X^{(t)}$. Then the following equations are trivial:

$$\sum_{X^{(t)} \in f_R^{(t)}(X_N)} m_R\left(X^{(t)}\right) = 2^N, \qquad \sum_{i=0}^{7} n_R\left(i, X^{(t)}\right) = N. \tag{2.10}$$

Using these parameters, the appearance probability of each configuration at the tth time step is obtained by

$$P_R^{(t)}(i) = \frac{1}{2^N \cdot N} \sum_{X^{(t)} \in f_R^{(t)}\left(\{X\}_N\right)} m_R\left(X^{(t)}\right) n_R\left(i, X^{(t)}\right). \tag{2.11}$$

Then the average weight of a configuration $X^{(t+1)}$ is given by

$$\left\langle w_R\left(X^{(t+1)}\right)\right\rangle_N = N\sum_{i=0}^{7} P_R^{(t)}(i) f_R^{(1)}(i)\,. \tag{2.12}$$

Modifying the rule function $f_R^{(1)}$, we define the tth *characteristic function* $\tilde{f}_R^{(t)}$. The following conditions are imposed on it:

(i) The mapping $\tilde{\boldsymbol{f}}_R^{(t)}$ constructed from $\tilde{f}_R^{(t)}$ is defined on $\{X\}_N$.

(ii) The value of a characteristic function is limited between 0 and 1:

$$0 \leq \tilde{f}_R^{(t)}(i) \leq 1 \quad \text{for} \quad i = 0, 1, \ldots, 7. \tag{2.13}$$

(iii) The average weight of a configuration contained in the image of the mapping $\tilde{\boldsymbol{f}}_R^{(t)}$ is equal to the above average weight of $X^{(t+1)}$.

The first condition means that the appearance probability $P_R(i)$ is equal to $\frac{1}{8}$. According to the second, a characteristic function cannot remain in CAs anymore. The image of $\tilde{\boldsymbol{f}}_R^{(t)}$ is a subset of the unit N-cube. As we discuss in later, a probability interpretation must be given. Although the images $\boldsymbol{f}_R^{(t+1)}(\{X\}_N)$ and $\tilde{\boldsymbol{f}}_R^{(t)}(\{X\}_N)$ are generally not identical, we must define $\tilde{f}_R^{(t)}$ as the third condition is satisfied. From these conditions, we have

$$\left\langle w_R\left(X^{(t+1)}\right)\right\rangle_N = \frac{N}{8}\sum_{i=0}^{7} \tilde{f}_R^{(t)}(i)\,. \tag{2.14}$$

Now we propose the following definition of a characteristic function:

$$\tilde{f}_R^{(t)}(i) = \begin{cases} f_R(i) & \text{for} \quad P_R(i) \geq \frac{1}{8}, \\ 8P_R(i) f_R(i) + \dfrac{\sum_{j \text{ for } P_R(j)\geq\frac{1}{8}} \left(P_R(j)-\frac{1}{8}\right) f_R(j)}{\sum_{j \text{ for } P_R(j)\geq\frac{1}{8}} \left(P_R(j)-\frac{1}{8}\right)} \left(1 - 8P_R(i)\right) & \\ & \text{for} \quad P_R(i) < \frac{1}{8}. \end{cases} \tag{2.15}$$

When $t = 0$, $\tilde{f}_R^{(0)}$ is identical with the rule function $f_R^{(1)}$ because there is no effect of the time evolution. Let us define a function space $\left\{\tilde{f} | \tilde{f} : \{X\}_3 \to [0,1]\right\}$. Then the characteristic function and all rules of the elementary CAs, from

rule 0 to rule 255, are contained in it. If a product of elements are defined by

$$\left(\tilde{f} \bullet \tilde{g}\right) \equiv \prod_{i=0}^{7} \left(\tilde{f}\,(i) \cdot \tilde{g}\,(i) + (1 - \tilde{f}\,(i)) \cdot (1 - \tilde{g}\,(i))\right), \tag{2.16}$$

the characteristic function can be expanded by the elementary CA rule functions as follows:

$$\tilde{f}_R^{(t)} = \sum_{r=0}^{255} \left(f_r \bullet \tilde{f}_R^{(t)}\right) f_r. \tag{2.17}$$

The coefficient $\left(f_r \bullet \tilde{f}_R^{(t)}\right)$ means a rate of change from the original rule R to rule r after t time evolutions. We call this expanded form as a *characteristic representation* of the rule function. In order to show adequacy of this representation, we calculated the coefficients for all independent even number rules of the elementary CAs by computer simulation. Some typical examples are listed in Table 2. All Class I rules change to rule 0. Most of Class II rules change to itself or typical Class II rules, say 2, 4, 34. Class III rules almost stay in the same Class. It is remarkable, however, that the Class IV rules 54 and 110 have larger changing rate to Class II rules than Class III rules (Table 3). These results indicate that their complex structures come from the interaction between aspects of Class II and III rules. Repetitions of limited patterns as the aspect of Class II are affected by chaotic variations as the aspect of Class III, and then complex patterns, soliton-like behaviors for example, are generated. In fact, as we will show in the next section, such Class IV-like patterns can be generated by characteristic functions of a linear combination of Class II and III rule functions. Now we conclude that the Class IV rules exist between Class II and III rules, that is, the so-called *edge of chaos*.

3 Discussion

Because the tth rule function $f^{(t)}$ of the elementary CAs is determined by 2^{2t+1} values, it is difficult to obtain its exact algebraic form after many time steps in general. Instead, we considered the characteristic representation which took into account the effects of the time evolution not directly but through changes of the appearance probabilities of the 3 sites configurations. It has an expansion form by all rule functions so that each coefficient means

a changing rate from the original rule to the corresponding one. This representation makes clear features of Class IV rules of the elementary CAs as well as can be used to classify the CA rules. As noted in Section 2, the calculated results showed that the changing rates from Class IV rules to Class II rules are larger than the rates from Class III rules to Class II rules. Using the characteristic representations, we can generate interference patterns such as Class IV rules. If we regard our representation as an extension of CAs to a stochastic model, each value of the characteristic function $\tilde{f}(x, y, z)$ can be interpreted as the probability that the site y will have a value 1 at the next time step. For example, we take such functions that

$$\tilde{f} = \frac{1}{2}f_4 + \frac{1}{2}f_{90}, \qquad \tilde{g} = \frac{1}{2}f_{232} + \frac{1}{2}f_{90}, \tag{3.18}$$

where rules 4, 232 and 90 are typical rules of Class II and III, respectively. These functions generate complex patterns as Fig. 2 which are analogous to the patterns of Class IV rules. Such interference patterns support the conjecture presented in Section 1, that is to say, features of Class IV rules comes from interference between the aspects of Class II and III rules.

Away from the theme of the classification of CAs, this dynamical system is very interesting. A pattern construction process can be shown by a continuous change of proper elements of $\tilde{f}(x, y, z)$. For example, rule 46 changes to rule 110 by varying the value of $\tilde{f}(1, 1, 0)$ from 0 to 1(Fig. 3). Moreover, this formalism will be able to be applied to more complex systems such as 5-neighbor CAs (now in preparation).

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